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Geometric Properties for a Class of Multivalent Convex Functions Using a Bessel-Based Integral Operator

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Abstract: **This This study pioneers the investigation into a novel class of multivalent convex functions of order η, defined within the unit disk and constructed via an integral operator originating from the generalized Bessel equation. We developed a systematic methodology to derive the initial coefficients, subsequently employing advanced complex analysis to establish a series of sharp and foundational results. Crucially, we determined the precise upper bounds (Coefficient Bounds), which are essential for understanding the function's growth limitations and series expansion. Furthermore, we proved Growth and Distortion theorems to delineate the exact geometric behavior and variability of these functions, which is vital for mapping and applications in complex analysis.**

One of the most important aspects in this study regarding the defined class is extracting a complete description of the extreme points. Identifying these points is paramount, as it facilitates the representation of any function in the class as a convex combination of these fundamental elements, thereby fully describing the geometric hull and confirming the sharpness of the established inequalities.

Keywords: **Bessel function; Coefficient bounds; Distortion Theorem, Growth Theorem and Integral operator.**

I. INTRODUCTION

Let $\mathcal{V}_p$ referring to the class of all p-valent analytic functions $g(w)$ in unit disk $\mathfrak{D} = \{w : w \in \mathbb{C}: \ |w| < 1\}$ that admit the following series representation:

$$g(w) = w^p + \sum_{k=1}^{\infty} c_{p+k} w^{p+k}, \qquad p \in \mathbb{N}\sim\{1\}. \tag{1}$$

Consider two functions $g(w)$ and $\mathscr{k}(w)$ are analytic within a domain $\mathfrak{D}$, the function $g(w)$ exhibits subordination to $\mathscr{k}(w)$ in $\mathfrak{D}$, denoted $g(w) \prec \chi(w)$, if there exists an analytic mapping $\sigma(w): \mathfrak{D} \to \mathfrak{D}$ satisfying:

- $\sigma(0) = 0$,
- $|\sigma(w)| < 1$ for all $w \in \mathfrak{D}$,

such that $g(w) = \chi(\sigma(w))$ holds identically. When $g(w)$ is p-valent in $\mathfrak{D}$, the subordination $g(w) \prec \chi(w)$ is equivalent to the conjunction of:

1. Initial value consistency: $g(0) = \mathscr{k}(0)$,
2. Range inclusion: The image of $g(\mathfrak{D})$ is contained within the image of $\chi(\mathfrak{D})$.

This equivalence is established [1].

In his foundational work [2], [3] and [4], Baricz introduced the generalized Bessel function(GBF) of order p of the first kind, defined by the power series

$$B(w) = B_{p,r,q}(w) = \sum_{m\geq 0} \frac{(-1)^m q^m}{m!\Gamma\left(p+m+\frac{r+1}{2}\right)} \left(\frac{w}{2}\right)^{2m+p}, \quad (2)$$

where $r, p, q, w \in \mathbb{C}$ and $q \neq 0$ and Γ stands for the Euler-Gamma function.

By applying ratio test, radius of convergence of series is measured for $B_{p,r,q}(w)$ is infinite; hence, the function be entire-it converges for all complex values of its parameters. This general formulation specializes to several classical functions, including the Bessel, spherical Bessel, modified Bessel and spherical modified Bessel functions of the first kind.

Furthermore, $B_{p,r,q}(w)$ satisfies the second-order linear differential equation

$$w^2 B''(w) + b\, wB'(w) + (qw^2 - p^2 + (1-r)p)B(w) = 0 \quad (3)$$

which is known as the generalized Bessel equation.

To investigate its univalence in the unit disk, a normalized form of the function defined in (3) is often employed. This normalization allows for the application of established results from geometric function theory.

In the present study, we consider the transformed function $u_{p,r,q}(w)$ defined by

$$h(w) = 2^m \Gamma\left(\mathrm{p} + \frac{\mathrm{r}+1}{2}\right) w^{-m}\, B_{p,r,q}(w), \quad (4)$$

where $B_{p,r,q}(w)$ is as given in (2).

By employing the standard Pochhammer symbol $(a)_m$ which defined for complex parameters a and m is defined via the Euler Gamma function as follows:

$$(a)_m = \frac{\Gamma(\mathrm{a}+\mathrm{m})}{\Gamma(\mathrm{a})} = \begin{cases} 1, & m = 0, a \in \mathbb{C}\backslash\{0\}, \\ a(a+1)\ldots(a+m-1), & m \neq 0, a \in \mathbb{C}, \end{cases}$$

also known as the rising factorial, the Pochhammer symbol $(a)_m$ has been extensively studied [5],[6],[7] and [8] and closely associated literature.

We get from the function $u_{p,r,q}(w)$ the next representation

$$h(w) = \sum_{m=0}^{\infty} \frac{(-q)^m}{m!\,(\mathcal{L})_m} \left(\frac{w}{2}\right)^{m+p}, \quad (5)$$

notably, this function is analytic across the complex plane and fulfills a second-order linear differential equation

$$w^2 h''(w) + r\, w\, h'(w) + (qw^2 - p^2 + (1-r)p)h(w) = 0.$$

Now, set

$$h(w) = \sum_{m=0}^{\infty} d_{m+p} \left(\frac{w}{2}\right)^{m+p}, \quad (6)$$

for all $w \in \mathbb{C}$, where

$$d_{m+p} = \sum_{m=0}^{\infty} \frac{(-q)^m}{m!\,(\mathcal{L})_m},$$

for all $m \geq 0$. Normalising (6) we have

$$h(w) = \left(\frac{w}{2}\right)^p + \sum_{k=1}^{\infty} d_{p+k} \left(\frac{w}{2}\right)^{k+p}, \quad (7)$$

For all $p \in \mathbb{N}$. Which it is equivalently

$$h(w) = \left(\frac{w}{2}\right)^p + \sum_{j=p+k}^{\infty} d_j \left(\frac{w}{2}\right)^j, \qquad k,p \in \mathbb{N} \tag{8}$$

The function defined in (8) known as the p-valent Bessel function, is analytic and multivalent in $\mathfrak{D}$.

Let $\mathcal{V}_{p,j}$ referred to the class of all functions which has the form (8). This class of functions, termed p-valent Bessel function [9].

In GFT, integral operators serve as a fundamental tool for constructing and analyzing subclasses of univalent functions. By applying such operators, one can transform existing functions into novel forms endowed with advantageous geometric properties-including convexity, starlikeness and close-to-convexity-especially within the unit disc. While certain operators of this type have been examined [10], the present study focuses on a novel integral operator introduced [11], defined as follows:

Let $\mathcal{T}_{\psi,\delta}^{\lambda} : \mathcal{V}_{p,j} \to \mathcal{V}_{p,j}$ such that

$$\mathcal{T}_{\psi,\delta}^{\lambda} h(w) = \frac{\psi+1}{z^{\psi}} \int_0^w w^{\psi-\delta} h(w)^{\delta} h'(w)^{\lambda} dw, \qquad \lambda,\delta \in \mathbb{R}, \psi \geq 0$$

$$\mathcal{T}_{\psi,\delta}^{\lambda} h(w) = \mathfrak{R}(w) = \frac{p^{\lambda}(\psi+1)}{\psi+p}\left(\frac{w}{2}\right)^p + \sum_{j=k+p}^{\infty} \frac{p^{\lambda-1}(\psi+1)(\delta p+\lambda j)}{\psi+j} d_j \left(\frac{w}{2}\right)^j, \qquad k,p \in \mathbb{N} \tag{9}$$

$$= \frac{p^{\lambda}(\psi+1)}{\psi+p}\left(\frac{w}{2}\right)^p + \sum_{j=k+p}^{\infty} \ell_j d_j \left(\frac{w}{2}\right)^j,$$

where $\ell_j = \frac{p^{\lambda-1}(\psi+1)(\delta p+\lambda j)}{\psi+j}$.

Previous studies have examined integral operators that incorporate GBF of order p of the first kind, analyzing their univalence properties. For a comprehensive review [12], [13], [14] and [15].

Building upon these studies, this paper establishes and investigates two novel subclasses of analytic functions. These classes are defined via an integral operator constructed from the Bessel function $\mathfrak{R}(w)$, given in (9).

Definition 1. Let λ, δ be real numbers, $\psi \in [0,\infty), t \in \mathbb{C}\backslash\{0\}$. Then a function $\mathfrak{R}(w)$ satisfies the membership condition for the class $\mathcal{SN}_{\psi,\delta}^{\lambda}(t,\chi(w))$ if and only if it meets the following requirement

$$\frac{1}{t}\left(1 + \frac{2\left(\frac{w}{2}\right)(\mathfrak{R}(w))''}{(\mathfrak{R}(w))'}\right) \prec \chi(w).$$

Definition 2. Let a function $\mathfrak{R}(w)$ satisfies the membership condition for the class $\mathcal{SN}_{\psi,\delta}^{\lambda}(t,w,\eta)$ of order η, if it meets the next requirement

$$Re\left(\frac{1}{t}\left(1 + \frac{2\left(\frac{w}{2}\right)(\mathfrak{R}(w))''}{(\mathfrak{R}(w))'}\right)\right) > \eta,$$

where $\chi(w) = \frac{1+(1-2\eta)w}{(1-w)}, 0 \leq \eta < 1, t \in \mathbb{C}\backslash\{0\}$ and all $w \in \mathfrak{D}$.

The univalent and multivalent functions theories are a focus of complex analysis and derive the complex mappings associated with a given branch of the complex plane. One of the important problems in this branch of mathematics is the characterization of the coefficients of Taylor series by precise sharp bounds. The characterization of these bounds is beyond the application of approximations of bounds; the conditions of these bounds reveal important properties of the functions which are analytic.

The fine structure of multivalent functions is given by the coefficients of the functions which compose the branching diagrams that place the virtual or physical branch points in the extended complex plane. The inequalities which are true for these branch points also apply to univalent functions, so in either case the coefficient bounds are the most useful information. In many domains, the encouragement of an ideal set of bounds leads researchers to the underlying theories of conformal maps, complex schemes, or Riemann surface structures.

The analytic bounds in the branch of mathematics comprising of allocated sets of coefficients with other bounds is the area that best balances the geometric context of representation with the analytic precision which clarifies the behavior of these functions. The candidate approaches to control the practical and theoretical manipulation of these functions also become more sophisticated and organized.

Theorem 1. If $\Re(w) \in \mathcal{SN}_{\psi,\delta}^{\lambda}(t, w, \eta)$, then

$$\sum_{j=k+p}^{\infty} \frac{j(\delta p + \lambda j)}{p(\psi + j)} |d_j| \leq \frac{p}{(\psi + p)}. \tag{10}$$

Proof: Let

$$G(w) = \frac{1}{t}\left(1 + \frac{2\left(\frac{w}{2}\right)(\Re(w))''}{(\Re(w))'}\right) - \eta$$

$$= \frac{(\Re(w))' + 2\left(\frac{w}{2}\right)(\Re(w))'' - t\,\eta(\Re(w))'}{t(\Re(w))'}$$

By applying requirement of the definition

$$G(w) \prec \frac{1+w}{1-w}$$

There is exist $\sigma(w)$ a Schwarz function, such that

$$G(w) = \frac{1+\sigma(w)}{1-\sigma(w)}.$$

This implies that

$$\sigma(w) = \frac{G(w)-1}{G(w)+1}.$$

We know that

$$|\sigma(w)| = \left|\frac{G(w)-1}{G(w)+1}\right| < 1.$$

Then

$$\left|\frac{G(w)-1}{G(w)+1}\right| = \left|\frac{2\left(\frac{w}{2}\right)(\Re(w))'' + (1 - t\,\eta - t)(\Re(w))'}{2\left(\frac{w}{2}\right)(\Re(w))'' + (1 - t\,\eta + t)(\Re(w))'}\right|$$

$$= \left|\frac{\frac{p^{\lambda+1}(p-1)(\psi+1)}{2(\psi+p)}\left(\frac{w}{2}\right)^{p-1} + \sum_{j=k+p}^{\infty} j(j-1)\ell_j d_j \left(\frac{w}{2}\right)^{j-1} + (1 - t\,\eta - t)\frac{p^{\lambda+1}(\psi+1)}{2(\psi+p)}\left(\frac{w}{2}\right)^{p-1} + \sum_{j=k+p}^{\infty}(1 - t\,\eta - t) j \ell_j\, d_j \left(\frac{w}{2}\right)^{j-1}}{\frac{p^{\lambda+1}(p-1)(\psi+1)}{2(\psi+p)}\left(\frac{w}{2}\right)^{p-1} + \sum_{j=k+p}^{\infty} j(j-1)\ell_j d_j \left(\frac{w}{2}\right)^{j-1} + (1 - t\,\eta + t)\frac{p^{\lambda+1}(\psi+1)}{2(\psi+p)}\left(\frac{w}{2}\right)^{p-1} + \sum_{j=k+p}^{\infty}(1 - t\,\eta - t) j \ell_j d_j \left(\frac{w}{2}\right)^{j-1}}\right|$$

$$= \left|\frac{\frac{(p - t\,\eta - t)}{2(\psi+p)} p^{\lambda+1}(\psi+1)\left(\frac{w}{2}\right)^{p-1} + \sum_{j=k+p}^{\infty}\frac{(j - t\,\eta - t)}{2}\, j\ell_j\, d_j \left(\frac{w}{2}\right)^{j-p}}{\frac{(p - t\,\eta + t)}{2(\psi+p)} p^{\lambda+1}(\psi+1)\left(\frac{w}{2}\right)^{p-1} + \sum_{j=k+p}^{\infty}\frac{(j - t\,\eta + t)}{2}\, j\ell_j\, d_j \left(\frac{w}{2}\right)^{j-p}}\right|$$

$$= \left| \frac{\frac{(p-t\eta-t)}{2(\psi+p)} p^{\lambda+1}(\psi+1) + \sum_{j=k+p}^{\infty} \frac{(j-t\eta-t)}{2} j\ell_j\, d_j \left(\frac{w}{2}\right)^{j-p}}{\frac{(p-t\eta+t)}{2(\psi+p)} p^{\lambda+1}(\psi+1) + \sum_{j=k+p}^{\infty} \frac{(j-t\eta+t)}{2} j\ell_j\, d_j \left(\frac{w}{2}\right)^{j-p}} \right|$$

$$\leq \frac{\frac{|p-t\eta-t|}{2(\psi+p)} p^{\lambda+1}(\psi+1) + \sum_{j=k+p}^{\infty} \frac{|j-t\eta-t|}{2} j\ell_j \quad |d_j| \quad \left|\frac{w}{2}\right|^{j-p}}{\frac{|p-t\eta+t|}{2(\psi+p)} p^{\lambda+1}(\psi+1) + \sum_{j=k+p}^{\infty} \frac{|j-t\eta+t|}{2} j\ell_j \quad |d_j| \quad \left|\frac{w}{2}\right|^{j-p}}.$$

The finally expression be bounded by number 1, if

$$\frac{|p-t\eta-t|}{2(\psi+p)} p^{\lambda+1}(\psi+1) + \sum_{j=k+p}^{\infty} \frac{|j-t\eta-t|}{2} j\ell_j \quad |d_j| \quad \left|\frac{w}{2}\right|^{j-p}$$

$$\leq \frac{|p-t\eta+t|}{2(\psi+p)} p^{\lambda+1}(\psi+1) + \sum_{j=k+p}^{\infty} \frac{|j-t\eta+t|}{2} j\ell_j \quad |d_j| \quad \left|\frac{w}{2}\right|^{j-p}$$

$$\sum_{j=k+p}^{\infty} j\ \ell_j |d_j| \ \leq \frac{p^{\lambda+1}(\psi+1)}{(\psi+p)}$$

Corollary 2. If $\Re(w) \in \mathcal{SN}_{\psi,\delta}^{\lambda}(t, w, \eta)$, then

$$|d_j| \leq \frac{p^2(\psi+j)}{j(\psi+p)(\delta p+\lambda j)}, \tag{11}$$

and equality in (11) holds for function $\Re(w)$, defined by

$$\Re(w) = \frac{p^{\lambda}(\psi+1)}{\psi+p}\left(\frac{w}{2}\right)^{p} + \frac{p^2(\psi+j)}{j(\psi+p)(\delta p+\lambda j)}\left(\frac{w}{2}\right)^{j}. \quad \blacksquare \tag{12}$$

The Growth and distortion theorems

The growth and distortion theorems concern univalent and multivalent functions and their behavior in the complex plane, and their geometric functions, both in the analytic and geometric realms of the complex plane, constituting a bounding framework within which their properties move and the mappings are preserved. It is, for instance, the growth theorem that an entire function which is bounded, has to be a constant function. This is one of the cornerstones in the analytic setting which under the boundedness have extreme restrictions to the entire functions of a complex variable.

On the other hand, the distortion theorem determines the analytic functions and their maximal flexible mechanics to be at curves and domains under mapping. It gives well-defined limits to the extent such functions can be dilation or contraction oriented to the geometric structures so that the deep splits between the functions in the analytics and the functions in the geometry are captured.

Eyeing these theorems, they are especially useful in understanding the more geometric analytic properties of the mappings of the complex plane which the behavior of the functions has to govern in the shape and the size.

Theorem 3. If $\Re(w) \in \mathcal{SN}_{\psi,\delta}^{\lambda}(t, w, \eta)$, then

$$\frac{p^{\lambda}(\psi+1)}{(\psi+p)} v^{p} - \frac{p^2(\psi+p+k)}{(p+k)(\psi+p)(\delta p+\lambda p+\lambda k)} v^{p+k} \leq |\Re(w)|$$

$$\leq \frac{p^{\lambda}(\psi+1)}{(\psi+p)} v^{p} + \frac{p^2(\psi+p+k)}{(p+k)(\psi+p)(\delta p+\lambda p+\lambda k)} v^{p+k} \tag{13}$$

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$\left|\frac{w}{2}\right| = \nu < 1$, provided $j \geq p+k$. This yields a sharp for

$\Re(w) = \frac{p^{\lambda}(\psi+1)}{(\psi+p)}\left(\frac{w}{2}\right)^{p} + \frac{p^{2}(\psi+p+k)}{(\psi+p)(p+k)(\delta p+\lambda p+\lambda k)}\left(\frac{w}{2}\right)^{p+k} \quad k,p \in \mathbb{N}$.

Proof: Applying inequality (10) valid for functions $\Re(w) \in \mathcal{SN}_{\psi,\delta}^{\lambda}(t,w,\eta)$ in conjunction with

$$\sum_{j=k+p}^{\infty} \frac{(p+k)(\delta p+\lambda p+\lambda k)}{p(\psi+p+k)} \leq \sum_{j=k+p}^{\infty} \frac{j(\delta p+\lambda j)}{p(\psi+j)}.$$

Then

$$\sum_{j=k+p}^{\infty} \frac{(p+k)(\delta p+\lambda p+\lambda k)}{p(\psi+p+k)} |d_j| \leq \sum_{j=k+p}^{\infty} \frac{j(\delta p+\lambda j)}{p(\psi+j)} |d_j| \leq \frac{p}{(\psi+p)}$$

$$\sum_{j=k+p}^{\infty} \frac{(p+k)(\delta p+\lambda p+\lambda k)}{p(\psi+p+k)} |d_j| \leq \frac{p}{(\psi+p)}$$

$$\sum_{j=k+p}^{\infty} |d_j| \leq \frac{p^2(\psi+p+k)}{(p+k)(\psi+p)(\delta p+\lambda p+\lambda k)} . \tag{14}$$

By using (14) for function $\Re(w) = \frac{p^{\lambda}(\psi+1)}{(\psi+p)}\left(\frac{w}{2}\right)^{p} + \sum_{j=k+p}^{\infty} d_j \left(\frac{w}{2}\right)^{j} \in \mathcal{SN}_{\psi,\delta}^{\lambda}(t,w,\eta)$, since $\left|\frac{w}{2}\right| = \nu$,

$$|\Re(w)| = \frac{p^{\lambda}(\psi+1)}{(\psi+p)}\nu^{p} + \sum_{j=k+p}^{\infty} d_j \ \nu^{j}$$

$$\leq \frac{p^{\lambda}(\psi+1)}{(\psi+p)}\nu^{p} + \nu^{k+p} \sum_{j=k+p}^{\infty} d_j$$

$$\leq \frac{p^{\lambda}(\psi+1)}{(\psi+p)}\nu^{p} + \frac{p^2(\psi+p+k)}{(p+k)(\psi+p)(\delta p+\lambda p+\lambda k)}\nu^{k+p} , \tag{15}$$

and similarly

$$|\Re(w)| \geq \frac{p^{\lambda}(\psi+1)}{(\psi+p)}\nu^{p} - \frac{p^2(\psi+p+k)}{(p+k)(\psi+p)(\delta p+\lambda p+\lambda k)}\nu^{k+p} . \quad \blacksquare \tag{16}$$

Theorem 4. If $\Re(w) \in \mathcal{SN}_{\psi,\delta}^{\lambda}(t,w,\eta)$, then

$$\frac{p^{\lambda+1}(\psi+1)}{2(\psi+p)}\nu^{p-1} + \frac{p^2(\psi+p+k)}{(\psi+p)(\delta p+\lambda p+\lambda k)}\nu^{k+p-1} \leq |\Re'(w)| \leq \frac{p^{\lambda+1}(\psi+1)}{2(\psi+p)}\nu^{p-1} + \frac{p^2(\psi+p+k)}{(\psi+p)(\delta p+\lambda p+\lambda k)}\nu^{k+p-1} \tag{17}$$

$\left|\frac{w}{2}\right| = \nu < 1$, provided $j \geq p+k$. Manifestly, this bound is sharp for

$\Re(w) = \frac{p^{\lambda}(\psi+1)}{(\psi+p)}\left(\frac{w}{2}\right)^{p} + \frac{p^{2}(\psi+p+k)}{(p+k)(\psi+p)(\delta p+\lambda p+\lambda k)}\left(\frac{w}{2}\right)^{p+k}$.

Proof: Application of the inequality show in (10) for $\Re(w) \in \mathcal{SN}_{\psi,\delta}^{\lambda}(t, w, \eta)$ then

$$\sum_{j=k+p}^{\infty} d_j \le \frac{p^2(\psi + p + k)}{(p+k)(\psi + p)(\delta p + \lambda p + \lambda k)}$$

$$\sum_{j=k+p}^{\infty} j d_j \le \frac{p^2(\psi + p + k)}{(\psi + p)(\delta p + \lambda p + \lambda k)}. \tag{18}$$

For function $\Re(w) = \frac{p^{\lambda}(\psi+1)}{(\psi+p)}\left(\frac{w}{2}\right)^p \sum_{j=k+p}^{\infty} d_j \left(\frac{w}{2}\right)^j$, then

$$|\Re'(w)| = \frac{p^{\lambda+1}(\psi+1)}{2(\psi+p)} v^{p-1} + \sum_{j=k+p}^{\infty} j d_j \ v^{j-1}$$

$$\le \frac{p^{\lambda+1}(\psi+1)}{2(\psi+p)} v^{p-1} + v^{p+k-1} \sum_{j=k+p}^{\infty} j d_j$$

$$\le \frac{p^{\lambda+1}(\psi+1)}{2(\psi+p)} v^{p-1} + \frac{p^2(\psi+p+k)}{(\psi+p)(\delta p + \lambda p + \lambda k)} v^{p+k-1}. \tag{19}$$

and similarly

$$|\Re'(w)| \ge \frac{p^{\lambda+1}(\psi+1)}{2(\psi+p)} v^{p-1} + \frac{p^2(\psi+p+k)}{(\psi+p)(\delta p + \lambda p + \lambda k)} v^{p+k-1}. \quad \blacksquare \tag{20}$$

Extreme points

When studying multivalent functions, identifying the extremal points is crucial. These points clarify the layout of the cut lines, pinpoint the essential singular points, and reveal the multiple branching behaviors that the function exhibits. Such investigations play a foundational role in unraveling the intricacies of the function itself and, consequently, in constructing the Riemann surface that the multivalent function automatically generates.

Theorem 5. Let $\Re_p(w) = \frac{p^{\lambda}(\psi+1)}{\psi+p}\left(\frac{w}{2}\right)^p$ and $\Re_j(w) = \frac{p^{\lambda}(\psi+1)}{\psi+p}\left(\frac{w}{2}\right)^p + \delta_j \frac{\frac{p}{(\psi+p)}}{M(\psi)} \left(\frac{w}{2}\right)^j$,

where $\Re(\psi) = \frac{j(\delta p + \lambda j)}{p(\psi + j)}$.

Then $\Re(w) \in \mathcal{SN}_{\psi,\delta}^{\lambda}(t, w, \eta)$, if it has the form

$\Re(w) = \delta_p \Re_p(w) + \sum_{j=k+p}^{\infty} \delta_j \Re_j(w)$, (21)

where $\delta_j \ge 0$ and $\delta_p = 1 - \sum_{j=k+p}^{\infty} \delta_j$.

Proof: Assume that

$\Re(w) = \delta_p \Re_p(w) + \sum_{j=k+p}^{\infty} \delta_j \Re_j(w)$.

$$= \left(1 - \sum_{j=k+p}^{\infty} \delta_j\right) \frac{p^{\lambda}(\psi+1)}{\psi+p}\left(\frac{w}{2}\right)^p + \sum_{j=k+p}^{\infty} \delta_j \left\{\frac{p^{\lambda}(\psi+1)}{\psi+p}\left(\frac{w}{2}\right)^p + \frac{\frac{p}{(\psi+p)}}{M(\psi)}\left(\frac{w}{2}\right)^j\right\}.$$

$$= \frac{p^{\lambda}(\psi+1)}{\psi+p}\left(\frac{w}{2}\right)^p + \sum_{j=k+p}^{\infty} \delta_j \frac{\frac{p}{(\psi+p)}}{M(\psi)}\left(\frac{w}{2}\right)^j$$

$$= \frac{p^{\lambda}(\psi+1)}{\psi+p}\left(\frac{w}{2}\right)^p + \sum_{j=k+p}^{\infty} d_j \left(\frac{w}{2}\right)^j.$$

Thus

$$\sum_{j=k+p}^{\infty} M(\psi)|d_j| = \sum_{j=k+p}^{\infty} M(\psi)\delta_j \frac{\frac{p}{(\psi+p)}}{M(\psi)}$$

$$= \frac{p}{(\psi+p)} \sum_{j=k+p}^{\infty} \delta_j$$

$$= \frac{p}{(\psi+p)}(1-\delta_p)$$

$$< \frac{p}{(\psi+p)},$$

From this we conclude

$\Re(w) \in \mathcal{SN}_{\psi,\delta}^{\lambda}(t,w,\eta)$.

By contrast. Think about this:

$$\Re(w) \in \mathcal{SN}_{\psi,\delta}^{\lambda}(t,w,\eta)$$

$|d_j| \leq \frac{\frac{p}{(\psi+p)}}{M(\psi)}, \qquad j = p+k, k \in \mathbb{N} \qquad .$

$$\delta_j \leq \frac{M(\psi)}{\frac{p}{(\psi+p)}} d_j, \qquad \delta_p = 1 - \sum_{j=k+p}^{\infty} \delta_j$$

Thus

$$\Re(w) = \frac{p^{\lambda}(\psi+1)}{(\psi+p)}\left(\frac{w}{2}\right)^p + \sum_{j=k+p}^{\infty} d_j \left(\frac{w}{2}\right)^j$$

$$\Re(w) = \left(\delta_p + \sum_{j=k+p}^{\infty} \delta_j\right)\frac{p^{\lambda}(\psi+1)}{\psi+p}\left(\frac{w}{2}\right)^p + \sum_{j=k+p}^{\infty} \delta_j \frac{\frac{p}{(\psi+p)}}{M(\psi)}\left(\frac{w}{2}\right)^j$$

$$= \delta_p \Re_p(w) + \sum_{j=k+p}^{\infty} \delta_j \left\{\frac{p^{\lambda}(\psi+1)}{\psi+p}\left(\frac{w}{2}\right)^p + \frac{\frac{p}{(\psi+p)}}{M(\psi)}\left(\frac{w}{2}\right)^j\right\}$$

$$= \delta_p \Re_p(w) + \sum_{j=k+p}^{\infty} \delta_j \ \Re_j(w). \qquad \blacksquare$$

II. CONCLUSION

This work investigates analytics of a new class of functions: multivalent convex functions of order η, constructed from an integral operator related to a generalized Bessel equation. This integral operator plays a central role and acts as a particular operator for an entire class of analytic functions with convexity conditions, and, hence, allows a unified treatment for a class of disparate analytic functions.

Following the developed systematic framework, we calculated, and proved to be valid the sharp bounds for the leading coefficients of their Taylor series. The bounds are rigorous in the sense that these are not asymptotic or abstract estimates, and can be illustrated with tangible examples, hence, are classified as hard bounds. Hence, the bounds enhance the analytic understanding of growth and optimal convex combinations, and, also, assist in the numerical inversion of Bessel type solutions.

The coefficient results were complemented with the proofs, the distortion and the growth theorems. They mathematically restrict the possible ranges of variation of the modulus, the foot-point distortion of the converse modulus, and the disparate behaviors of different analytic and coëfficient derivatives. Theorems demonstrate that such explicit estimates greatly aid in function's domain mapping and clarify the allowed bilinear growth with order-preserving mappings.

Above all, this work delivers an integral geometric and analytic guide to the extreme points of the new family, the stage where the risks decide the payoff and therefore the deciding proof set of the proofs. The recent work theorems become elementary, since any function within the family is expressed as the exact concentration of extreme points, and therefore the closed convex boundary of points becomes evident. Such compact representations sharpen the inequalities and so their extremality all grows rigid.

In summary, the study materializes a functional continuity between polyhedral aspects of boundary-solution theory and core aspects of geometric function theory. The order of work associates both houses. Hence the paper lays the functional table. The next phase appears as broadly divided. Two strands, perhaps, materialize: to investigate strata-closers convex-ring, to articulate family-openings focused under different geometrically-inherited linear operators, and to employ the convex as a technical core against concrete equations of theoretical physics.

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